

3. Kinematics in 1-D

Definition of velocity

If a particle covers the distance s between points P_1 and P_2 in time t ,

one can talk of the *average velocity* between P_1 and P_2 as:

$$v_{av} = s / t$$

or:

$$v_{av} = \frac{x(P_2) - x(P_1)}{t(P_2) - t(P_1)}$$

or:

$$v_{av} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

Does it make any sense to speak of the velocity at an *instant* of time, eg. at time t_1 ?

Clearly, as the time interval gets shorter (and hence the two points P_1 and P_2 get closer together), then v_{av} is a better estimate of the velocity at t_1 .

But what happens when the time interval goes to *zero*?
(Zeno's Paradox)

The answer to this question required the invention of a whole new mathematical tool - differential calculus (Newton and Leibnitz).

Mathematically,

$$v(t_1) = \lim_{\delta t \rightarrow 0} \frac{x(t + \delta t) - x(t)}{\delta t}$$

$$= \dot{x} \quad (\text{Newton})$$

$$= \frac{dx}{dt} \quad (\text{Leibnitz})$$

The operation is known as *differentiation* and follows well-known mathematical rules.

The inverse operation is known as *integration*.

$$x = \int v dt$$

BASIC MATHEMATICAL RESULTS

1. Differentiation

$$\frac{d}{dt} f(t) = \lim_{\delta t \rightarrow 0} \frac{f(t + \delta t) - f(t)}{\delta t}$$

graphical interpretation: the *slope* of the curve at point t

some results:

$$\frac{d}{dt} k = 0$$

$$\frac{d}{dt} t = 1 \quad (k = \text{constant})$$

$$\frac{d}{dt} t^2 = 2t$$

$$\frac{d}{dt} t^3 = 3t^2 \quad \dots \text{etc}$$

$$\frac{d}{dt} \sin \omega t = \omega \cos \omega t$$

$$\frac{d}{dt} \cos \omega t = -\omega \sin \omega t$$

2. Product rule for differentiation:

$$\frac{d}{dt} f(t)g(t) = f(t) \frac{d}{dt} g(t) + g(t) \frac{d}{dt} f(t)$$

NOTE: thus $\frac{d}{dt} kf(t) = k \frac{d}{dt} f(t)$

3. Integration

- Inverse of differentiation
- graphical interpretation: the area under a curve
- constant of integration

some results:

$$\int k dt = kt + c \quad \int kt dt = \frac{1}{2} kt^2 + c$$

Definition of acceleration

$$a = \dot{v} = \frac{dv}{dt}$$

$$a = \ddot{s} = \frac{d^2s}{dt^2}$$

Higher derivatives of position

While it is mathematically possible to define higher derivatives of position, this is not necessary in practice.

The reason is Newton's insight that force produces an acceleration. Thus, if the forces are known at all times, the resulting motion is uniquely determined by double-integrating the resulting acceleration.

The Laws of Physics as differential equations

Newton's greatest contribution to science was the concept that physical laws can be expressed as differential equations.

It is usually quite simple to write down a differential equation representing a given situation. The problem is solving it. (except in simple cases)

Mathematicians since Newton have developed methods of handling these equations
→ birth of theoretical physics

An alternative to analytical solution is the use of *numerical methods*, which today is greatly simplified by the availability of computers.

Example - Motion with constant acceleration

$$v = \int a \, dt$$

$$v = at + \text{constant}$$

$$v = u + at$$

let $v(t=0)=u$, so:

$$s = \int v \, dt$$

$$s = \int (u + at) \, dt$$

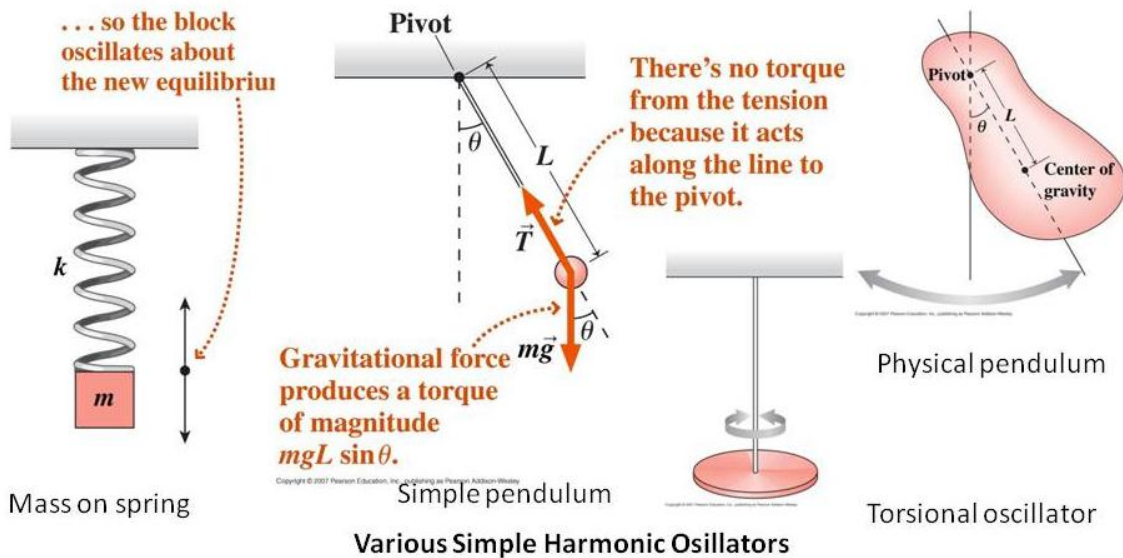
$$s = ut + \frac{1}{2} at^2$$

and as $s(t=0)=0$

Exercise: derive the equation:

$$v^2 = u^2 + 2as$$

Example - motion with non-constant acceleration



Consider the case of motion where:

$$a = -kx \quad (\text{Simple Harmonic Motion})$$

Then, a solution to this equation is:

$$x = A \cos(\omega t + \phi)$$

(Prove it if you have done A level Maths. Otherwise ignore)

- Meaning of A and ϕ : boundary conditions
- This may not be the only solution to this equation.
- Many equations cannot be solved so simply

Let us go back to our original (non-differential) definitions.

Suppose $x(0)$ and $v(0)$ are known.

Then if δt is small enough, we can write:

$$a(0) = -k x(0)$$

$$v(0 + \delta t) = v(0) + a(0) \cdot \delta t$$

and: $x(0 + \delta t) = x(0) + v(0) \cdot \delta t$

In fact, if $x(t)$, $v(t)$ and $a(t)$ are known:

$$v(t + \delta t) = v(t) + a(t) \cdot \delta t$$

and: $x(t + \delta t) = x(t) + v(t) \cdot \delta t$

and the value of x at any time t can be found by repeating this process a large number of times.

To increase the accuracy of the estimation, all that needs to be done is make δt smaller, at the cost of having to go through the loop more times.

Actually, the above process does not give very good results in practice. The problem is that the velocity is changing over the interval δt .

A better method is to use the *average* value of the velocity in the interval $(t, t + \delta t)$. This can be approximated as the velocity at the middle of the interval, ie. $v(t + \frac{1}{2} \delta t)$

Thus the method becomes:

To start:

$$a(0) = -k x(0)$$

$$v(0 + 1/2 \delta t) = v(0) + a(0) \cdot 1/2 \delta t$$

$$x(0 + \delta t) = x(0) + v(0 + 1/2 \delta t) \cdot \delta t$$

and the iteration is:

$$a(t) = -k x(t)$$

$$v(t + 1/2 \delta t) = v(t) + a(t) \cdot \delta t$$

$$x(t + \delta t) = x(t) + v(t + 1/2 \delta t) \cdot \delta t$$

Let's try it. (use a calculator)

Suppose:

$$k = 1$$

$$x(0) = 1$$

$$v(0) = 0$$

$$\delta t = 0.1$$

t	$x(t)$	$a(t)$	$v(t+1/2 \delta t)$	$x(t+\delta t)$
INITIAL CONDITIONS	1		0	

0	1	-1	-0.05	0.995
0.1	0.995	-0.995	-0.1495	0.98005
0.2	0.98005	-0.98005	-0.24751	0.952995
0.3	0.952995			
0.4	etc....			

How to do it with Excel:

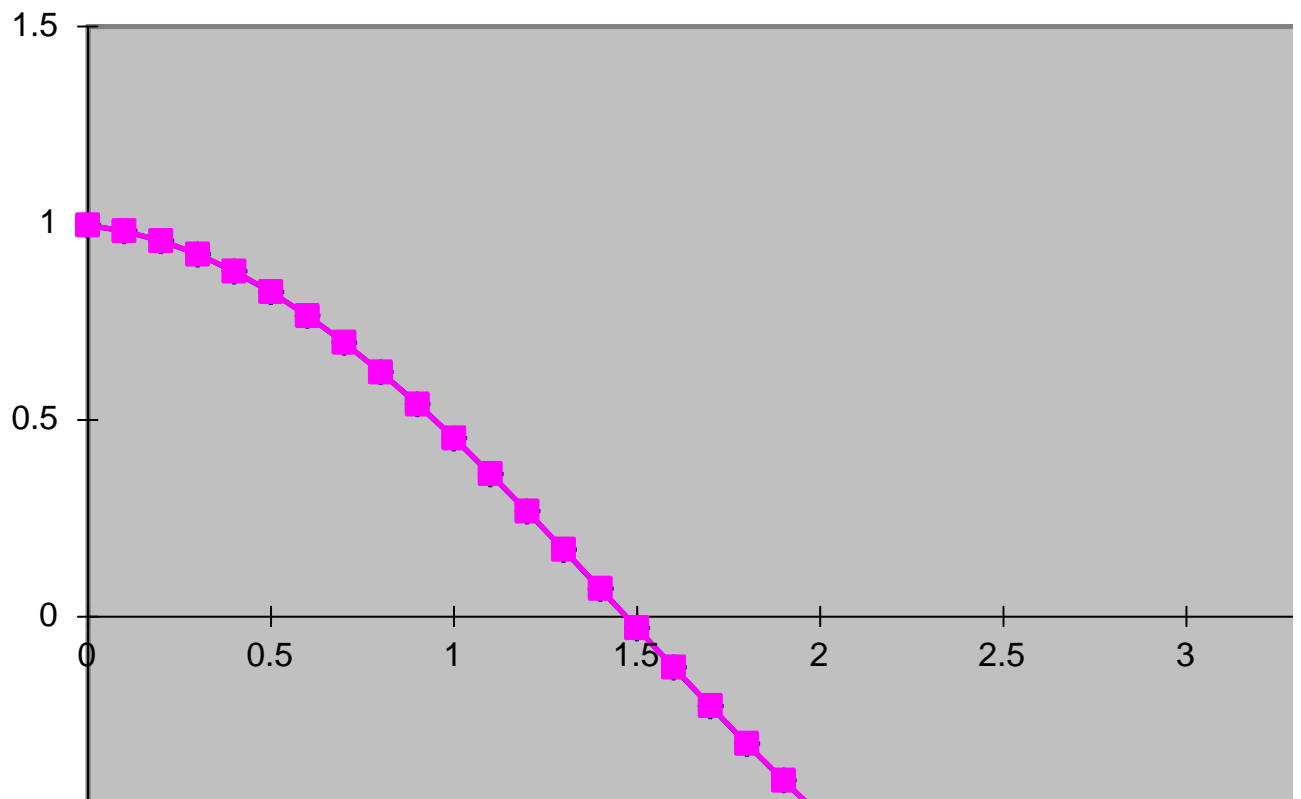
	A	B	C	D	E	F
1	INIT	0.1	1		0	

2	t	dt	x(t)	a(t)	v (t+dt/2)	x(t+dt)
3	0	=B1	=C1	= - C3	=E1+ D3*B3/2	=C3+ E3*B3
4	=A3+B3	=B3	=F3	= - C4	=E3+ D4*B4	=C3+ E4*B4
5	=A4+B4	=B4	=F4	= - C5	=E4+ D5*B5	=C4+ E5*B5

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INIT	0.1	1		0		
t	dt	x(t)	a(t)	v(t+dt/2)	x(t+dt)	cos(t+dt)
0	0.1	1	-1	-0.05	0.995	0.995004
0.1	0.1	0.995	-0.995	-0.1495	0.98005	0.980067
0.2	0.1	0.98005	-0.98005	-0.24751	0.9553	0.955336
0.3	0.1	0.9553	-0.9553	-0.34303	0.920996	0.921061
0.4	0.1	0.920996	-0.921	-0.43513	0.877483	0.877583
0.5	0.1	0.877483	-0.87748	-0.52288	0.825194	0.825336
0.6	0.1	0.825194	-0.82519	-0.6054	0.764654	0.764842
0.7	0.1	0.764654	-0.76465	-0.68187	0.696467	0.696707
0.8	0.1	0.696467	-0.69647	-0.75151	0.621316	0.62161

0.9	0.1	0.621316	-0.62132	-0.81365	0.539951	0.540302
1	0.1	0.539951	-0.53995	-0.86764	0.453187	0.453596
1.1	0.1	0.453187	-0.45319	-0.91296	0.361891	0.362358
1.2	0.1	0.361891	-0.36189	-0.94915	0.266976	0.267499
1.3	0.1	0.266976	-0.26698	-0.97585	0.169392	0.169967
1.4	0.1	0.169392	-0.16939	-0.99279	0.070113	0.070737
1.5	0.1	0.070113	-0.07011	-0.9998	-0.02987	-0.0292
1.6	0.1	-0.02987	0.029867	-0.99681	-0.12955	-0.12884
1.7	0.1	-0.12955	0.129548	-0.98386	-0.22793	-0.2272
1.8	0.1	-0.22793	0.227933	-0.96106	-0.32404	-0.32329
1.9	0.1	-0.32404	0.324039	-0.92866	-0.41691	-0.41615
2	0.1	-0.41691	0.416905	-0.88697	-0.5056	-0.50485
2.1	0.1	-0.5056	0.505602	-0.83641	-0.58924	-0.5885
2.2	0.1	-0.58924	0.589243	-0.77748	-0.66699	-0.66628
2.3	0.1	-0.66699	0.666991	-0.71078	-0.73807	-0.73739
2.4	0.1	-0.73807	0.73807	-0.63698	-0.80177	-0.80114
2.5	0.1	-0.80177	0.801767	-0.5568	-0.85745	-0.85689
2.6	0.1	-0.85745	0.857447	-0.47106	-0.90455	-0.90407
2.7	0.1	-0.90455	0.904553	-0.3806	-0.94261	-0.94222
2.8	0.1	-0.94261	0.942613	-0.28634	-0.97125	-0.97096
2.9	0.1	-0.97125	0.971247	-0.18921	-0.99017	-0.98999
3	0.1	-0.99017	0.990168	-0.0902	-0.99919	-0.99914
3.1	0.1	-0.99919	0.999188	0.009721	-0.99822	-0.99829
3.2	0.1	-0.99822	0.998216	0.109543	-0.98726	-0.98748
3.3	0.1	-0.98726	0.987262	0.208269	-0.96643	-0.9668
3.4	0.1	-0.96643	0.966435	0.304912	-0.93594	-0.93646
3.5	0.1	-0.93594	0.935944	0.398507	-0.89609	-0.89676
3.6	0.1	-0.89609	0.896093	0.488116	-0.84728	-0.8481
3.7	0.1	-0.84728	0.847281	0.572844	-0.79	-0.79097
3.8	0.1	-0.79	0.789997	0.651844	-0.72481	-0.72593
3.9	0.1	-0.72481	0.724812	0.724325	-0.65238	-0.65364



Extensions:

- Make the delta-time interval smaller (more rows!)
- Use a programming language rather than Excel
- Use higher mathematical precision (more significant figures)
- Go to 2 or 3 dimensions (work out each dimension separately in each interval)
- Have more than one moving object eg: two or more planets interacting by gravity

etc.....