WAVES (cont'd)

SUPERPOSITION OF WAVES

When two waves:

$$y_1 = A_1 \cos(\omega_1 t - k_1 x)$$
$$y_2 = A_2 \cos(\omega_2 t - k_2 x)$$

pass through the same area, the resultant displacement at any point and at any time is given by the sum of the two individual displacements:

$$y = y_1 + y_2$$

This is known as the ***principle of superposition***

Superposition:

- only holds in linear media

- lets waves "pass through" each other without affecting one another except in the region of overlap

- is responsible for many interesting phenomena eg. interference, standing waves, Fourier synthesis

- in *non-linear* media where superposition does not hold, other interesting effects occur, eg. wave mixing and harmonic generation.

Example: harmonic generation and mixing in non-linear media

Suppose a wave travels in a medium where the force is between particles is some complicated function of the displacement.

This function can be expanded as a power series:

$$F(x) = k_1 x + k_2 x^2 + \ldots$$

where the term in x represents elastic (linear) behaviour and the higher terms represents departures from linearity.

We have already considered the linear case. Suppose now that the force is proportional to the *square* of the displacement

$$F = ky^2$$

If the wave has the usual form $y = A \cos(\omega t)$, then the force has the form:

$$\begin{aligned} F &= kA^2 \cos^2(\omega t) \\ &= \tfrac{1}{2} kA^2 [1 + \cos(2\omega t)] \end{aligned}$$

which has twice the frequency of the original wave

A force dependent on the *cube* of the displacement will contain a component in the third harmonic (3ω), and so on.

To take a more complicated example, suppose two waves:

$$y_1 = A_1 \cos(\omega_1 t)$$

$$y_2 = A_2 \cos(\omega_2 t)$$

pass through our square-law medium. Then the force is:

$$\begin{aligned} F &= kA^2 (y_1 + y_2)^2 \\ &= kA^2 \left[\cos^2(\omega_1 t) + \cos^2(\omega_2 t) + 2\cos(\omega_1 \omega_2 t)\right] \end{aligned}$$

The first 2 terms are the same as in the previous case (second harmonic generation): however there is now a cross term in the *product* of the two waves.

The product term can be expanded:

$$\cos(\omega_1 \omega_2 t) = \tfrac{1}{2}\left[\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t\right]$$

ie. The output contains components with the *sum* and *difference* of the two frequencies. This is termed *mixing*.

Applications:

- tuning of radio receivers by conversion to an *intermediate frequency*
- *optical up-conversion* to obtain blue light from a red laser

Fourier Syntheisis

Mathematics students are all familiar with polynomial expansions, whereby functions having certain “well-behaved” properties can be expressed as an infinite sum of terms of increasing powers.

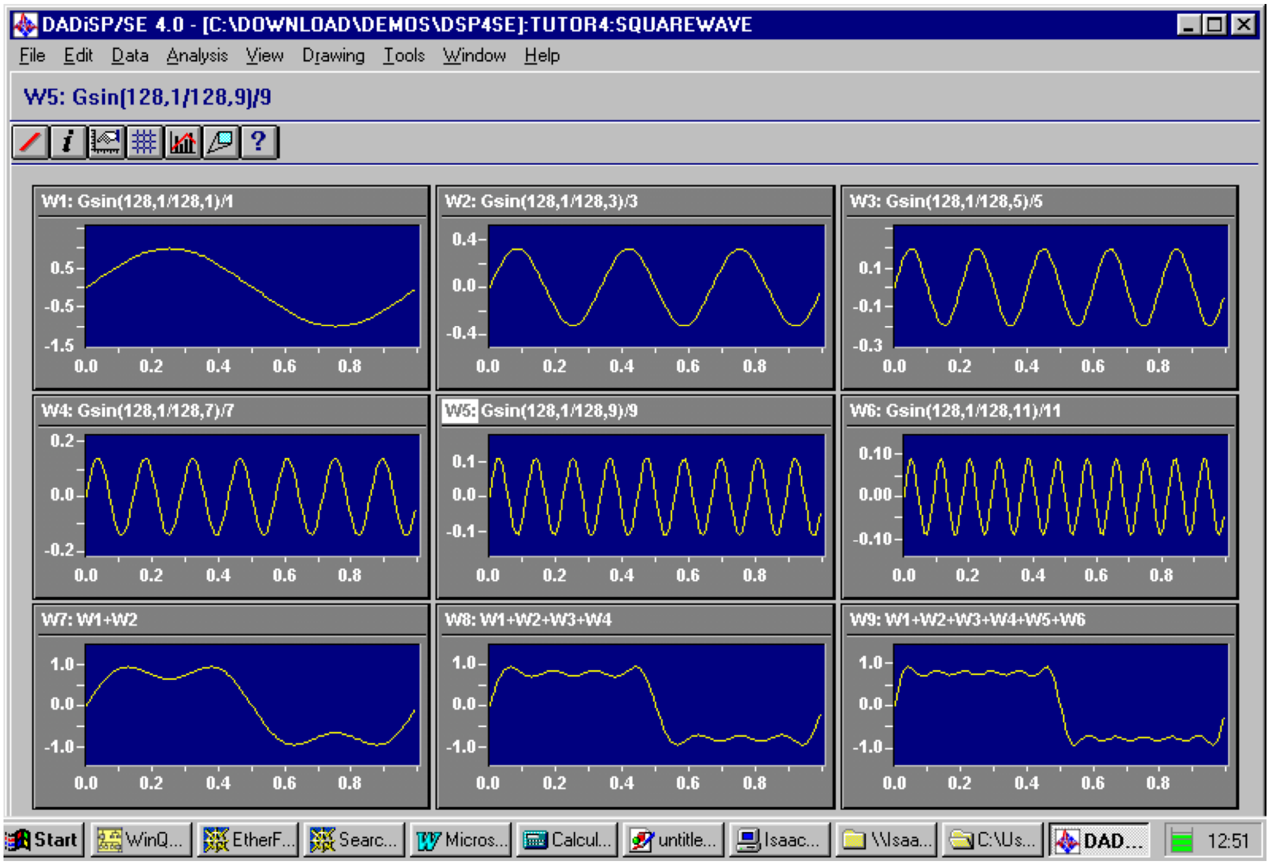
$$f(t) = \sum A_n t^n$$

A very useful and similar theorem is due to Fourier, and states that any *repetitive* function (that is also “well-behaved”) can be expressed as an infinite sum of *sines and cosines* of increasing frequency:

$$f(t) = \sum A_n \cos n\omega t + B_n \sin n\omega t$$

where the period of repetition of the function is $2\pi/\omega$.

Example: Fourier synthesis of a square wave $f(t) = \sum_{n=1,3,5...} \frac{1}{n} \sin n\omega t$



Note: Fourier's Theorem is stated without proof. For an interactive exercise in its use, load the "Fourier" program in the "Albert" suite available in the Computing Lab.

Some more notes on the Fourier Transform:

- it is a *transform* in that it is used to transform a function from one domain to another. In mathematical terms, it transforms from the *time domain* to the *frequency domain*.

- Modern digital signal processing makes extensive use of the Fourier Transform since many operations are simpler to perform in the frequency domain than in the time domain.

- Fourier synthesis was used in early electronic musical instruments to produce different waveforms by mixing together various harmonics of a fundamental - hence the name *synthesiser*

- Its use is not only restricted to electronics - it crops up all over the place. The frequency spectrum of *any* wave is the Fourier transform of its waveform. For example, the pattern of bright and dark fringes in a hologram (or any interference pattern) are the Fourier transform of the object causing the interference pattern.

Interference

The principle of superposition implies that waves interfere with each other in the area where they overlap.

If the two waves have exactly the same frequency and phase relationship (*coherent)*, then the resultant interference pattern in space will be steady in time. The problem with seeing interference with light is that it is difficult to produce coherent light - unless a laser is used, of course.

Conversely, if the two waves have different frequencies, then the resultant summed wave will also have a time dependence. These are known as *beats*.

Beats

Suppose two waves of the same amplitude A and different frequencies ω_1 and ω_2 meet at the point $x=0$. Then we have:

$$y = A\left[\cos\omega_1 t + \cos\omega_2 t\right]$$

Let:

$$\varpi = \frac{\omega_1 + \omega_2}{2} \qquad \Delta\omega = \frac{\omega_1 - \omega_2}{2}$$

Then:

$$\omega_1 = \varpi - \Delta\omega \qquad \omega_2 = \varpi + \Delta\omega$$

Using the trigonometric identity:

$$\cos(a \mp b) = \cos a \cos b \pm \sin a \sin b$$

the above equation can be written:

$$y = 2A\sin\left(\Delta\omega\, t\right)\cos\varpi\, t$$

$$y = 2A \sin(\Delta\omega\, t) \cos \varpi\, t$$

This represents a wave with:

- twice the amplitude of the original waves (if they had different amplitudes, it would have been the sum);

- the average frequency of the two waves;

- modulated by an *amplitude envelope* that is half the difference between the two frequencies.

- In the case of sound, our ears are actually sensitive to the intensity of a wave or y^2. We therefore hear a *beat frequency* equal to the difference of the two frequencies.

Applications:

- tuning of musical instruments
- sound effects
- in electronics, obtaining the difference between two (possibly very high) frequencies, eg. in Doppler radar.