Lecture 8 & 9: Special Relativity

The two postulates of Special Relativity:

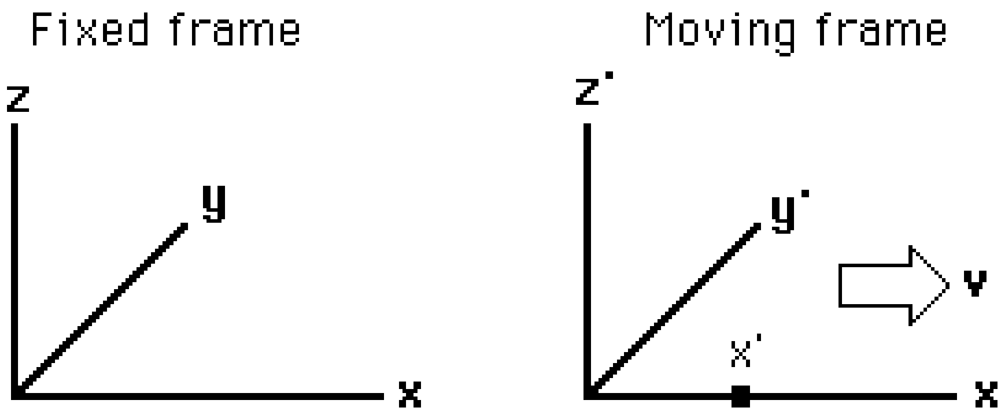
1. The speed of light is the same for all observers, no matter what their relative speeds.

2. The laws of physics are the same in any inertial (that is, non-accelerated) frame of reference.

Basic difference between the two theories of physics:

- Newtonian mechanics: uses **Galilean transformation**
- Special Relativity: **Lorentz transformation**

Lorentz transformation



The primed frame moves with velocity v in the x direction with respect to the fixed reference frame. The reference frames coincide at t=t'=0. The point x' is moving with the primed frame.

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The reverse transformation is:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

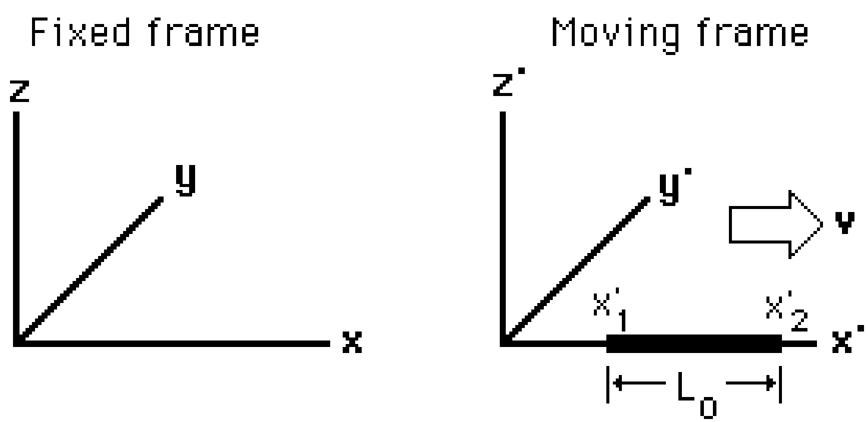
$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is the **Lorentz factor**

Length contraction



The length of any object in a moving frame will appear foreshortened in the direction of motion, or contracted. The amount of contraction can be calculated from the **Lorentz transformation**.

The length is maximum in the frame in which the object is at rest.

If the length $L_0 = x'_2 - x'_1$ is measured in the moving reference frame, then $L = x_2 - x_1$ can be calculated using the Lorentz transformation.

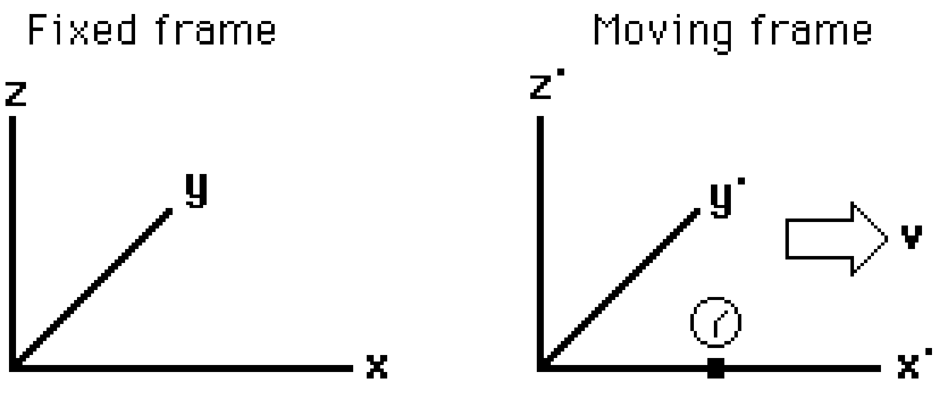
$$L_0 = x'_2 - x'_1 = \frac{x_2 - vt_2 - x_1 + vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

But since the two measurements made in the fixed frame are made simultaneously in that frame, $t_2 = t_1$, and

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma}$$

Length contraction

Time dilation



A clock in a moving frame will be seen to be running slow, or "dilated" according to the Lorentz transformation. **The time will always be shortest as measured in its rest frame**. The time measured in the frame in which the clock is at rest is called the **proper time**.

If the time interval $T_0 = t'_2 - t'_1$ is measured in the moving reference frame, then $T = t_2 - t_1$ can be calculated using the Lorentz transformation.

$$T = t_2 - t_1 = \frac{t'_2 + \frac{vx'_2}{c^2} - t'_1 - \frac{vx'_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The time measurements made in the moving frame are made at the same location, so the expression reduces to:

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} = T_0 \gamma$$

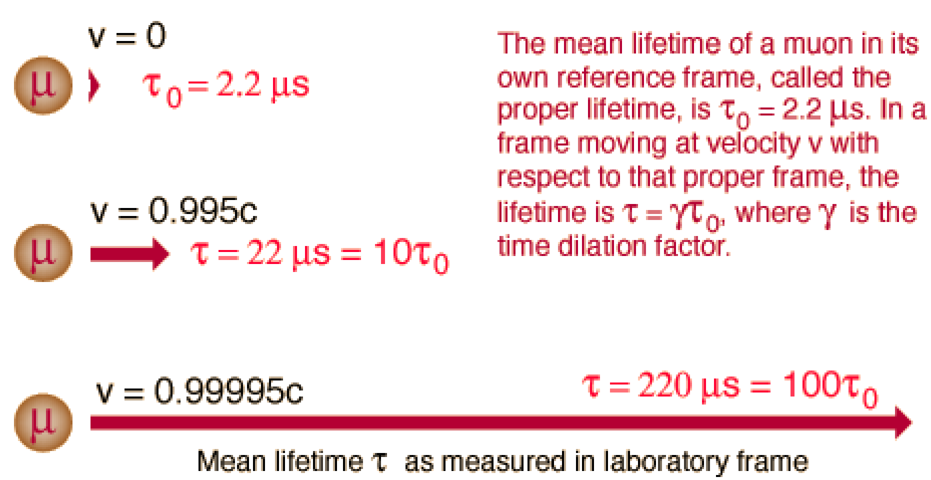
Time dilation experiments

Twin Paradox

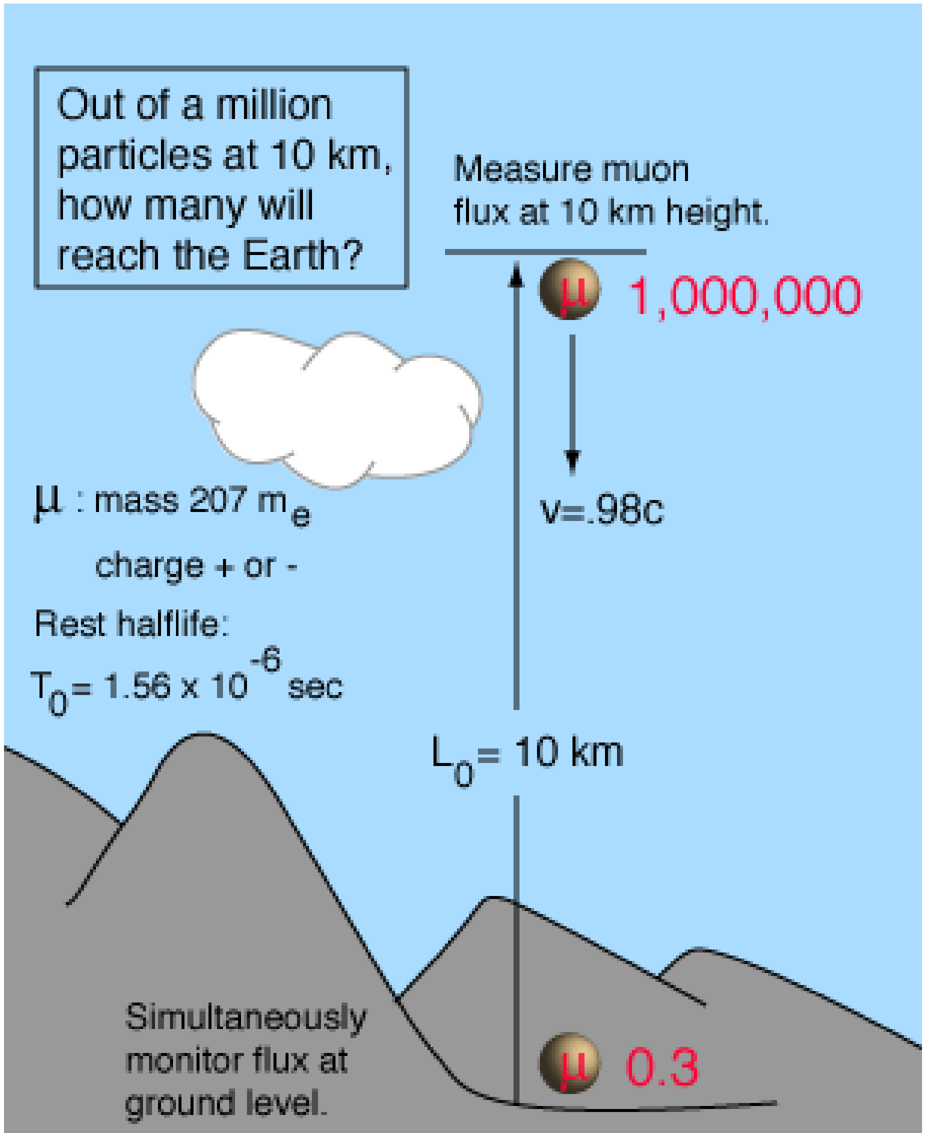
The story is that one of a pair of twins leaves on a high speed space journey during which he travels at a large fraction of the speed of light while the other remains on the Earth. Because of time dilation, time is running more slowly in the spacecraft as seen by the earthbound twin and the traveling twin will find that the earthbound twin will be older upon return from the journey. **The common question: Is this real? Would one twin really be younger?**

The basic question about whether time dilation is real is settled by the **muon experiment**. **The clear implication is that the traveling twin would indeed be younger**, but the scenario is complicated by the fact that the traveling twin must be accelerated up to traveling speed, turned around, and decelerated again upon return to Earth. Accelerations are outside the realm of Special Relativity and require **General Relativity**.

Despite the experimental difficulties, an experiment on a commercial airliner confirms the existence of a time difference between ground observers and a reference frame moving with respect to them.

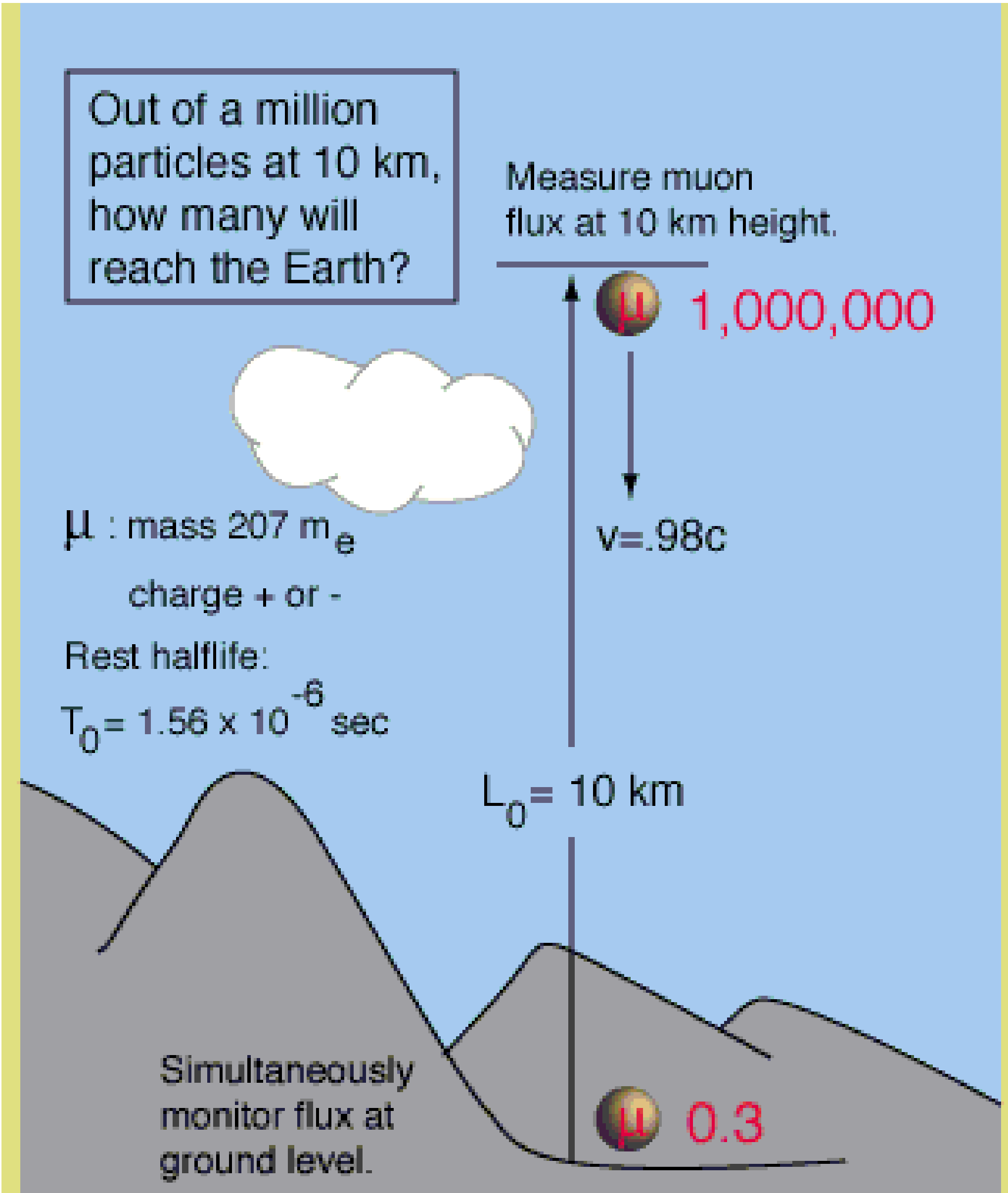


Muon experiment: Experimental verification of time dilation and length contraction



The measurement of the flux of muons at the Earth's surface produced an early dilemma because many more are detected than would be expected, based on their short half-life of 1.56 microseconds.

Non-Relativistic



Distance: $L_0 = 10^4$ meters

Time: $T = \dfrac{10^4\ \text{m}}{(0.98)(3 \times 10^8\ \text{m/s})}$

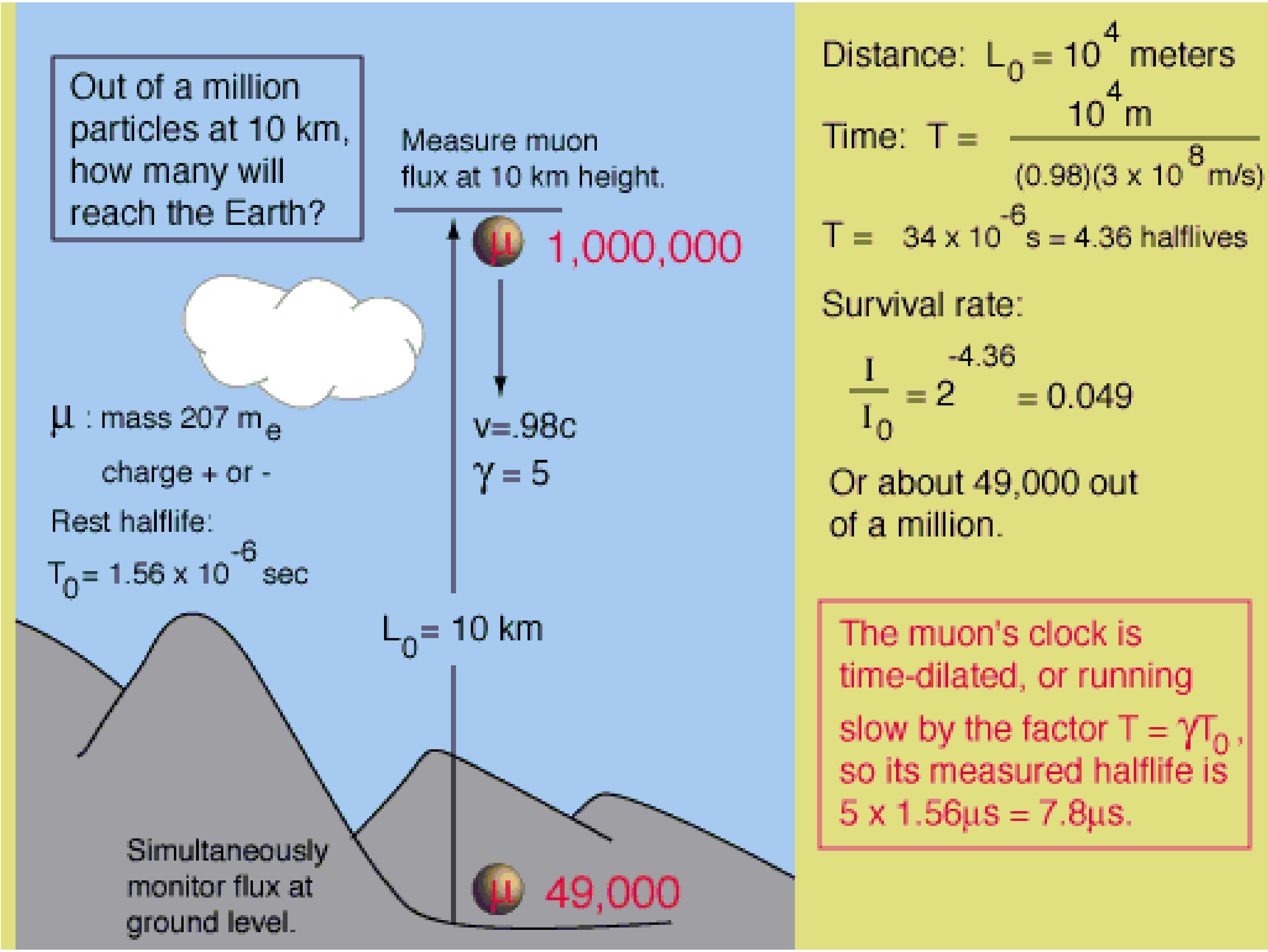
$T = 34 \times 10^{-6}\ \text{s} = 21.8$ halflives

Survival rate:

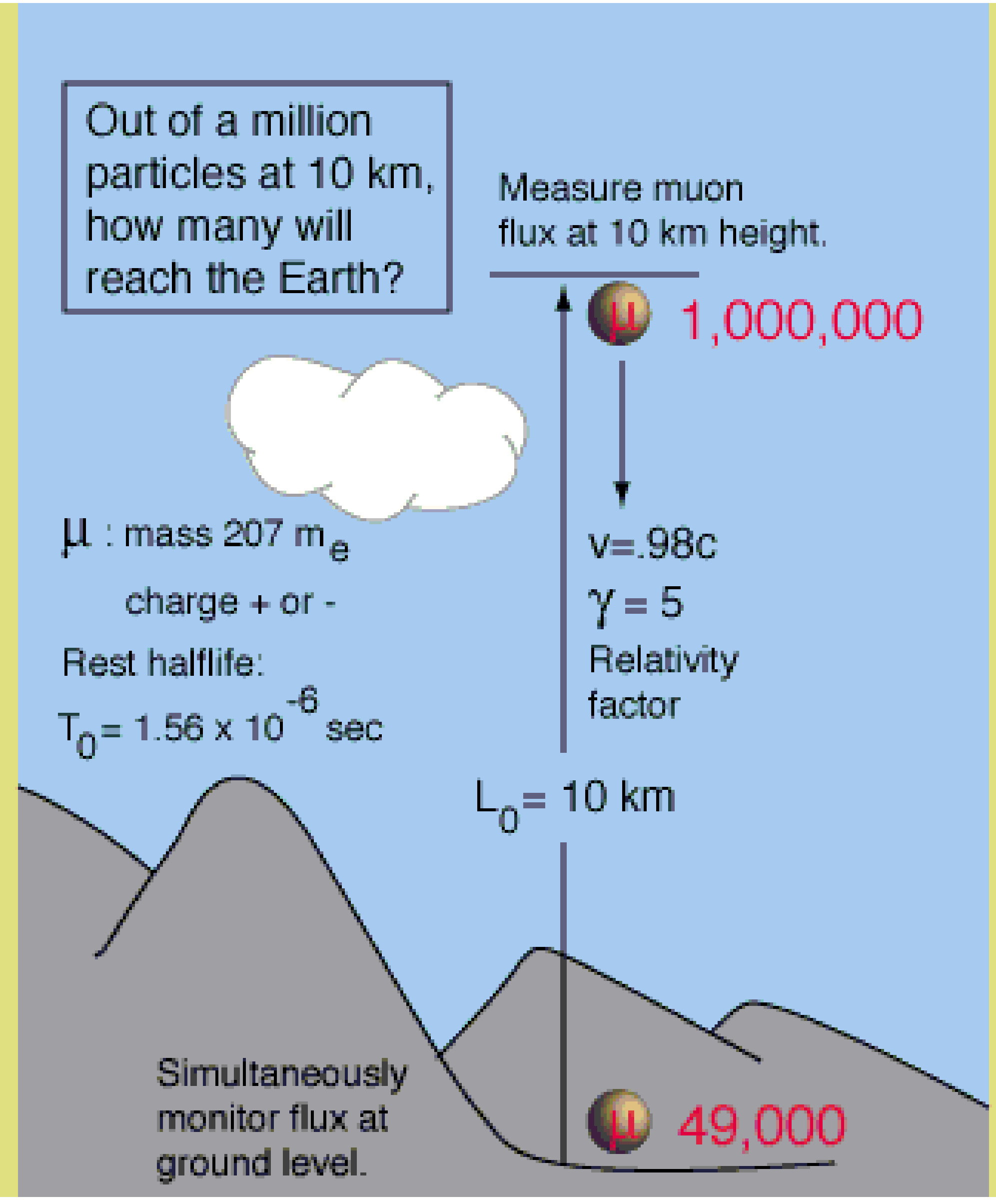
$$\frac{I}{I_0} = 2^{-21.8} = 0.27 \times 10^{-6}$$

Or only about 0.3 out of a million.

Relativistic, Earth-Frame Observer



Relativistic, Muon-Frame Observer



Distance: $L_0 = 10^4$ meters

Time: $T = \dfrac{2000 \text{ m}}{(0.98)(3 \times 10^8 \text{ m/s})}$

$T = 6.8 \times 10^{-6}$ s = 4.36 halflives

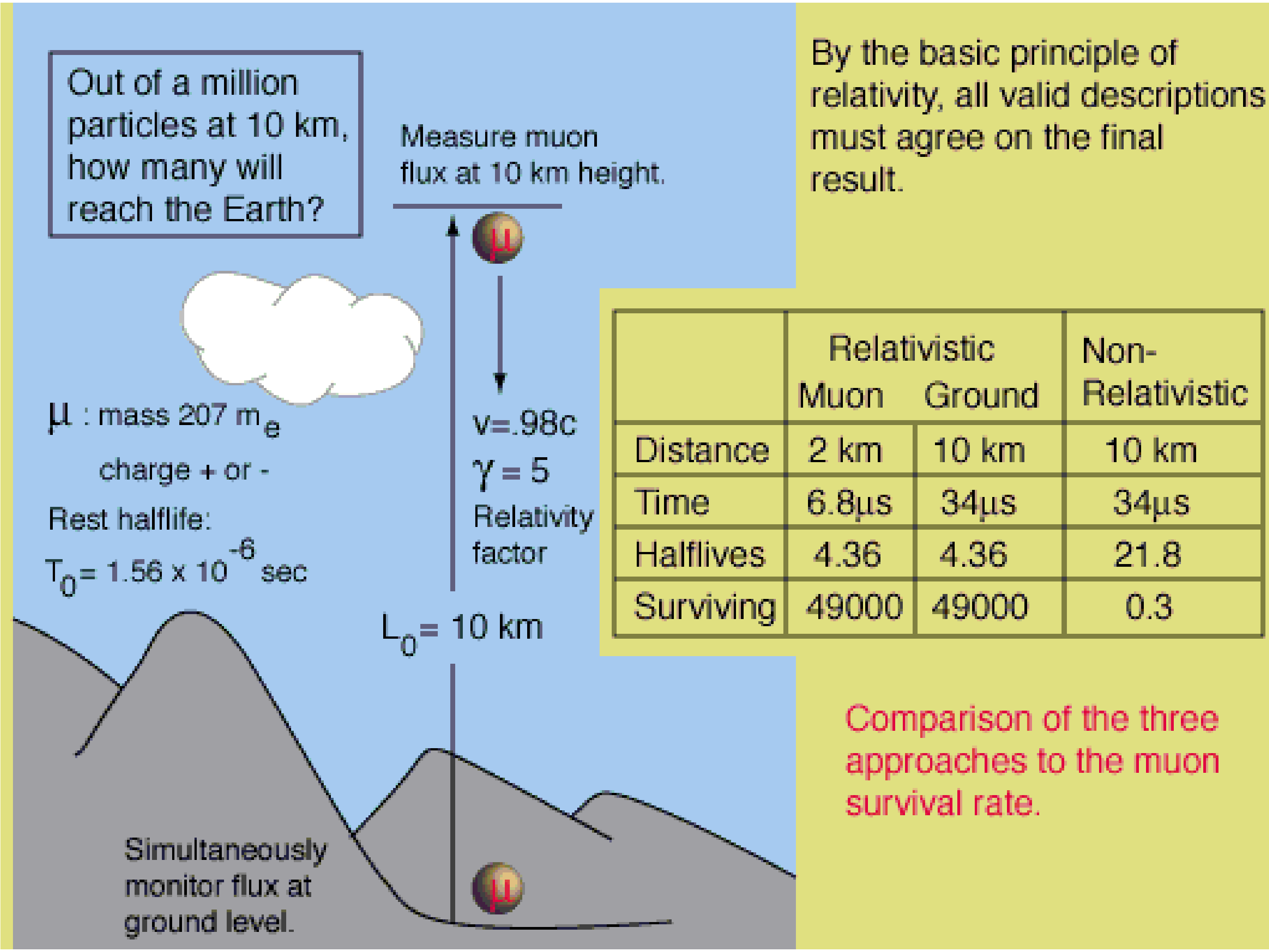
Survival rate:

$$\frac{I}{I_0} = 2^{-4.36} = 0.049$$

Or about 49,000 out of a million.

The muon sees distance as length-contracted so that $L = L_0/\gamma = 0.2L_0$ = 2 km.

Comparison of Reference Frames

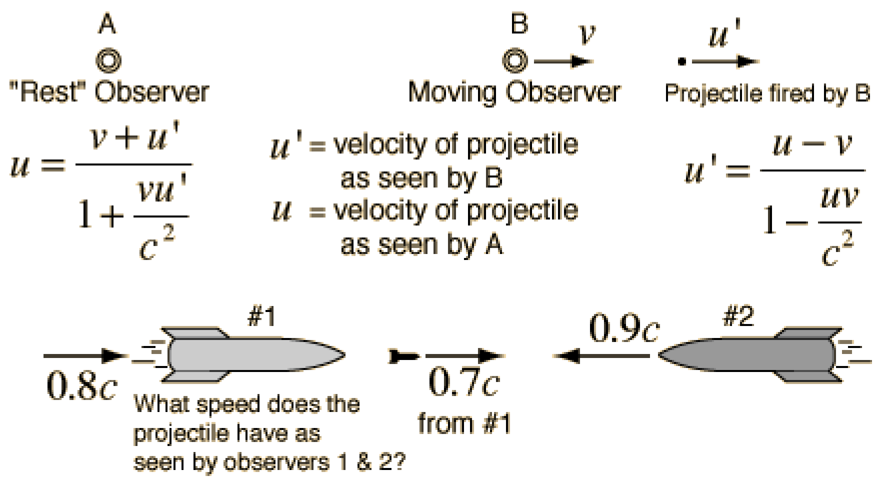


	Relativistic Muon	Relativistic Ground	Non-Relativistic
Distance	2 km	10 km	10 km
Time	6.8μs	34μs	34μs
Halflives	4.36	4.36	21.8
Surviving	49000	49000	0.3

In the muon experiment, the relativistic approach yields agreement with experiment and is greatly different from the non-relativistic result. **Note that the muon and ground frames do not agree on the distance and time, but they agree on the final result.** One observer sees time dilation, the other sees length contraction, but neither sees both.

Relative velocities

We have described how length and time change in Special Relativity. What about velocity (length/time)?

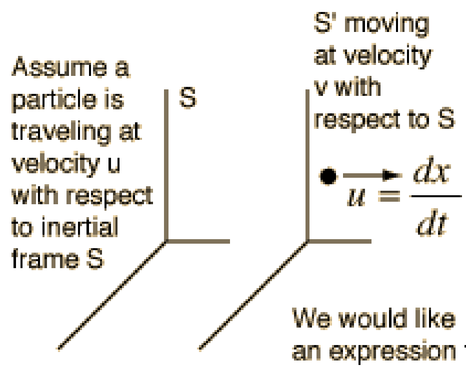


Galilean relativity says $v_{rel} = v' + v''$

But in the above situation, we would obtain velocities greatear than c. Therefore, we must modify the way we calculate relative velocity.

Relativistic Velocity Transformation

No two objects can have a relative velocity greater than c. But what if I observe a spacecraft traveling at 0.8c and it fires a projectile which it observes to be moving at 0.7c with respect to it? Velocities must transform according to the **Lorentz transformation**, and that leads to a very non-intuitiveresult called **Einstein velocity addition**.



Lorentz transformation

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{xv}{c^2})$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We would like an expression for $u' = \frac{dx'}{dt'}$ which is the velocity as measured in the moving reference frame S'

Just taking the derivatives of these quantities leads to the velocity transformation. For the Lorentz transformation expressions for x' and t' above gives

$$\frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{vdx}{c^2})} = \frac{\frac{dx}{dt} - v}{1 - \frac{v\frac{dx}{dt}}{c^2}}$$

Putting this in the notation introduced in the illustration above:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

The reverse transformation is obtained by just solving for u in the above expression. Doing that gives

$$u = \frac{u' + v}{1 + \frac{uv}{c^2}}$$

.

Relativistic Relative Velocity

The speed of light is the speed limit of the universe, so it follows that no observer will see any other observer approaching or receding at a speed greater than c. But what if observers A and B are both moving toward each other with speeds approaching c as seen by an external observer? How will A and B measure their relative speeds? This is an example of **Einstein velocity addition**. In the calculation below, velocities to the right are taken as positive.

Common sense suggests an approach speed = $|v_A| + |v_B|$

To apply the relativistic velocity relationship, we must identify the appropriate velocity to associate with each symbol.

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

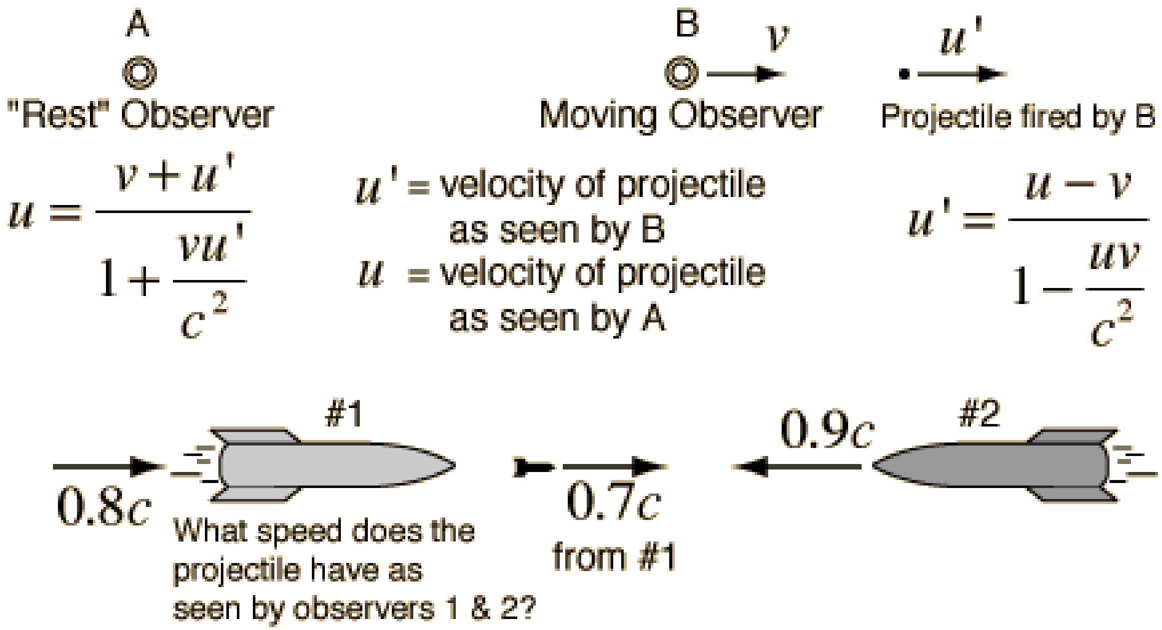
Let v_B be the velocity as seen by an external reference frame: $u = v_B$

Then v_A is the speed of the secondary reference frame: $v = v_A$

Then u' is the speed of B as seen by A (the relative velocity):

$$u' = \frac{v_B - v_A}{1 - \frac{v_B v_A}{c^2}}$$

Note that since v_B is negative, this reduces to the common sense version for $v_B v_A << c^2$.



Applying the Einstein velocity transformation to the spacecraft traveling at 0.8c which fires a projectile which it observes to be moving at 0.7c with respect to it, we obtain a velocity of 1.5c/1.56 = 0.96c rather than the 1.5c which seems to be the common sense answer.

Remember

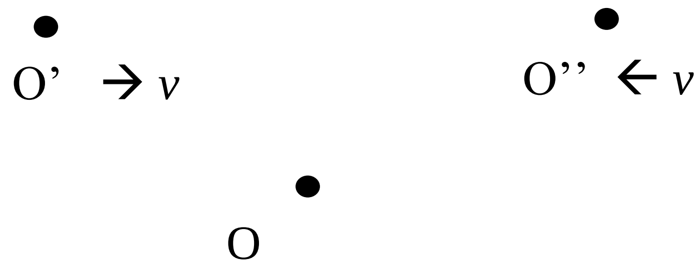
$$v_{rel} = \frac{v' + v''}{1 + \dfrac{v' v''}{c^2}}$$

Show that:

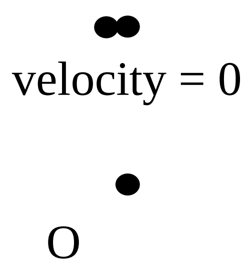
- The formula reduces to the Galilean equation for velocities much smaller than c
- If (say) $v' = v'' = 0.9\,c$, v_{rel} is still smaller than c
- If $v'' = c$, then $v_{rel} = c$ whatever the value of v'
- If $v' = v'' = c$, then v_{rel} is still c

Mass in Special Relativity

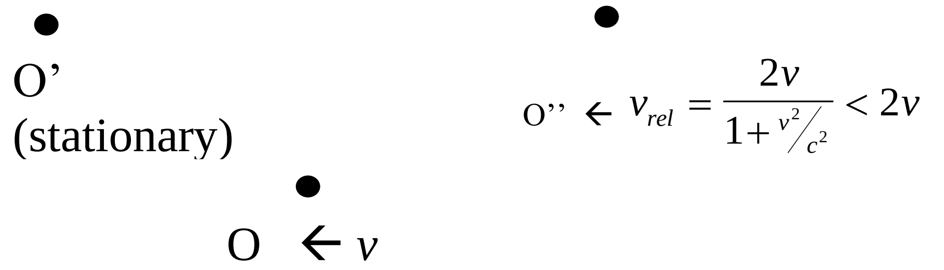
Suppose two identical particles O' and O'' , each of mass *m* , are approaching each other so that their velocities in the frame of an observer at rest O are
v and -*v* respectively.



If on collision the two particles stick together, they will finish at rest in the frame of O (since total momentum = 0).



What does the collision look like in the frame of O’ ?



Einstein kept the principle of **conservation of momentum**.

But using *p=mv* we obtain a different answer in different rest frames (the objects will *not* be at rest with respect to O).

We want to keep the formula for momentum.
We modified the definition of velocity *v.*
Therefore, we must also modify the definition of mass *m*.

Mass increase

We have seen that two observers in relative motion agree on their velocity relative to one another. Hence in two such frames, the magnitude of *v* is not in doubt.

$$p = \frac{m}{\sqrt{1 - v^2/c^2}}\, v \quad = (\gamma\, m) v$$

The above equation can be written:

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} \quad = \gamma (mv)$$

in other words:

$$p = mv \qquad \Rightarrow \qquad m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\, m_0 \qquad m_0 = \text{"rest mass"}$$

Therefore, conservation of momentum requires that the mass of an object increases with velocity.

m_0 is known as the *rest mass* of the object.

The mass increase gives another “explanation” of why it is impossible to travel faster than light. If a particle is accelerated by a constant force, Newton’s laws predict a constant acceleration to infinite velocity.

However, if the mass *increases* with velocity, then the acceleration will decrease as the velocity increases.

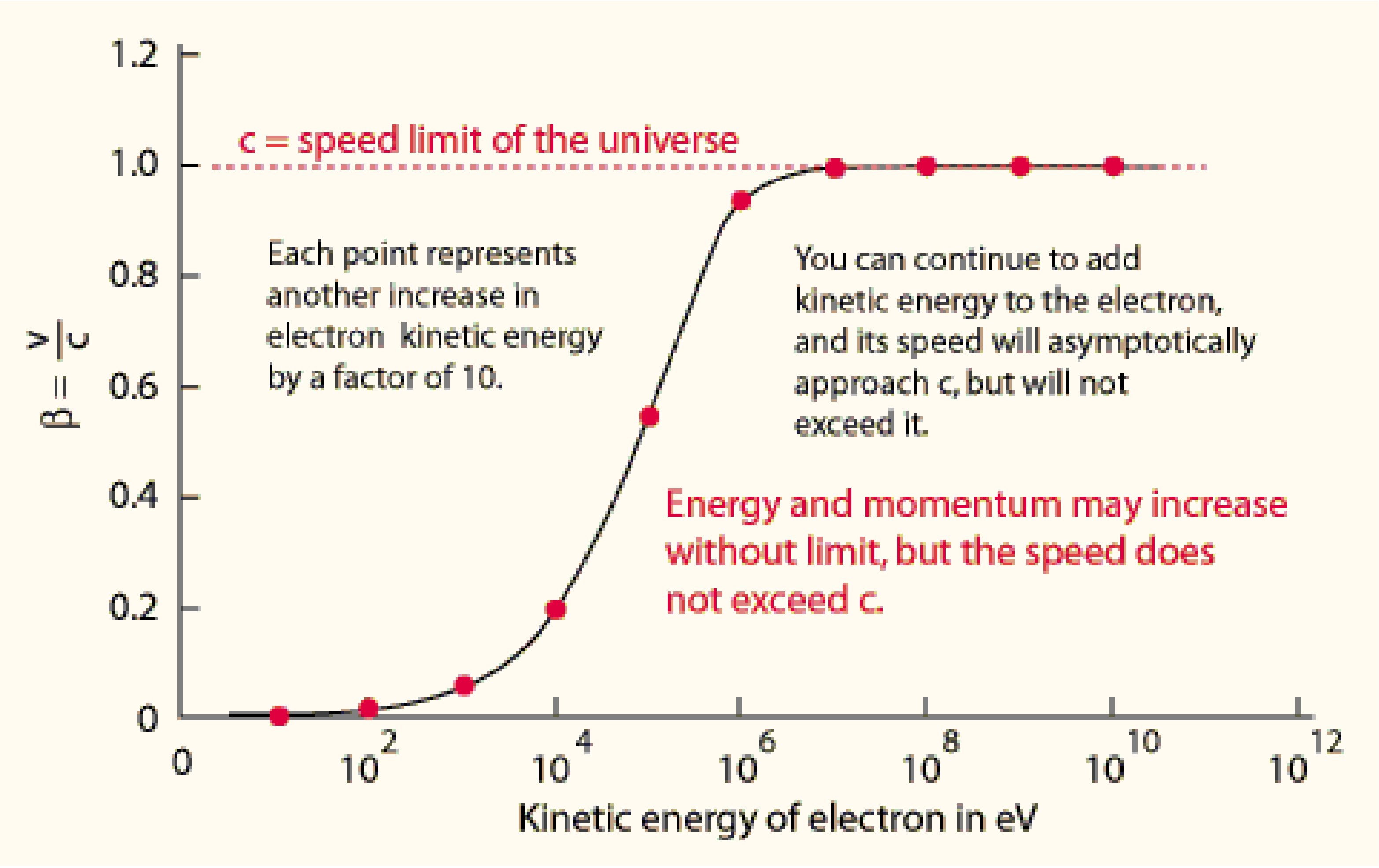
Another way of putting it is that it will require an infinite amount of kinetic energy $\frac{1}{2} mv^2$ for an object to travel at the speed of light, as m tends to infinity.

Why is it impossible to accelerate anything to a velocity exceeding c?

Mass increases as velocity approaches c.

Recall: Mass is the resistance of an object to acceleration.

Therefore, the more you accelerate the object, the greater the resistance to acceleration.



Relativistic energy

Einstein found that the **kinetic energy** of a moving body is

$$E_k = m_0(\gamma - 1)c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2,$$

He included the second term on the right to make sure that for small velocities, the energy would be the same as in classical mechanics:

$$E_k = \frac{1}{2} m_0 v^2 + \cdots$$

Without this second term, there would be an additional contribution in the energy when the particle is not moving.

The **total momentum (conserved in collisions)** of a moving particle is:

$$P = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

The relativistic mass and the relativistic kinetic energy are related by the formula:

$$E_k = mc^2 - m_0 c^2.$$

Einstein wanted to omit the unnatural second term on the right-hand side, whose only purpose is to make the energy at rest zero, and to declare that the particle has a total energy which obeys:

$$E = mc^2$$

$$mc^2 = m_0c^2 + T$$

where T is the kinetic energy. Thus we can write:

$$E = E_0 + T$$

In other words, we can consider the **total energy** of an object to be composed of the **energy equivalent of the rest mass** and the kinetic energy.

Alternatively,

$$m = m_0 + \frac{T}{c^2}$$

This implies that the mass of an object is composed of its rest mass and the **mass equivalent of the kinetic energy**.

Conservation of Energy

In the presence of a conservative force, the principle of conservation of mass-energy can be stated as follows:

$$mc^2 - (T + V) = m_0 c^2 \qquad \text{(constant)}$$

where V is the (relativistic) kinetic energy, V is the potential energy, and moc2 the "mass-energy" of the object at rest.

Relativity: summary

- ***c* is constant for all inertial observers**
- **time dilation** $t = t'\gamma$
- **length contraction** $x = x'\gamma$
- **mass increase** $m' = m\gamma$
- **mass-energy relation** $E=mc^2$

For an excellent summary, see http://hyperphysics.phy-astr.gsu.edu/hbase/relativ/conrel.html