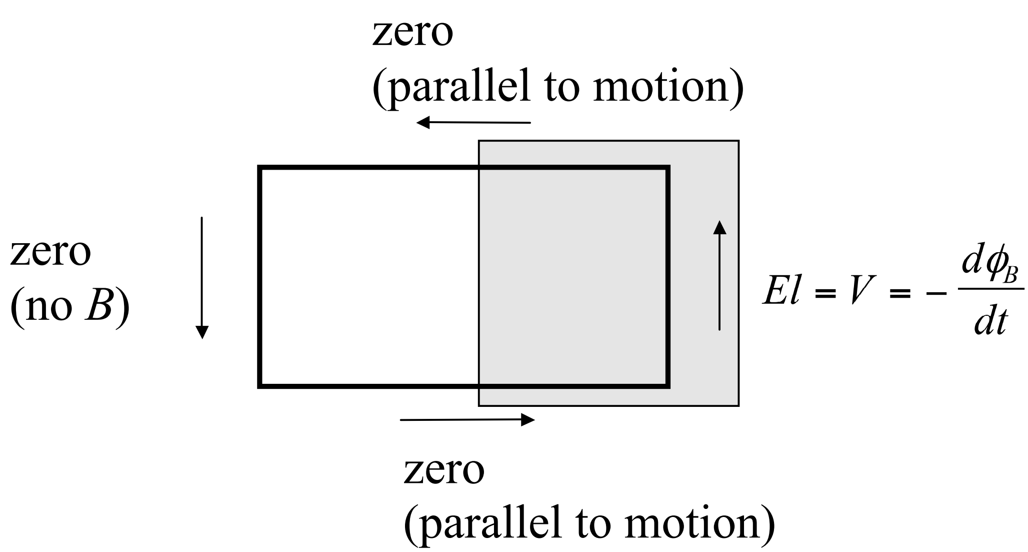
ELECTROMAGNETISM (cont'd)

11. Maxwell's Equations

Clearly, when we have electromagnetic induction, $\oint \mathbf{E}.d\mathbf{l} \neq 0$ (equation IIa)

The term $\oint \mathbf{E}.d\mathbf{l}$ means the product of **E** and **l**, summed round a complete loop. Let us do that for our example above:



Thus, we can write:

$$\oint \mathbf{E}.d\mathbf{l} = -\frac{d\phi_{\mathbf{B}}}{dt}$$

Now, the flux is just the integrated field through a surface, ie:

$$\phi_{\mathbf{B}} = \int \mathbf{B}.d\mathbf{S}$$

Hence,

$$\boxed{\oint \mathbf{E}.d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B}.d\mathbf{S}} \qquad \textbf{... II}$$

Similarly, taking equation (IIIa) presented earlier:

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0 I$$

and noting that $I = dq/dt$, where in turn from equation (I):

$$q = \varepsilon_0 \oint_S \mathbf{E}.d\mathbf{S}$$

we can write:

$$\boxed{\oint \mathbf{B}.d\mathbf{l} = \mu_0 \varepsilon_0 \frac{d}{dt}\oint \mathbf{E}.d\mathbf{S}} \qquad \textbf{... III}$$

We can now gather these 4 equations which form a complete description of electromagnetic phenomena in free space (without the complicating effects of matter):

$$\oint_S \mathbf{E}.d\mathbf{S} = \frac{q}{\varepsilon_0} \qquad \ldots \text{ I}$$

$$\oint \mathbf{E}.d\mathbf{l} = -\frac{d}{dt}\oint_S \mathbf{B}.d\mathbf{S} \qquad \ldots \text{ II}$$

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0 \varepsilon_0 \frac{d}{dt}\oint_S \mathbf{E}.d\mathbf{S} \qquad \ldots \text{ III}$$

$$\oint_S \mathbf{B}.d\mathbf{S} = 0 \qquad \ldots \text{ IV}$$

These equations were first enunciated by James Clerk Maxwell (1831-1879), and together with Newton's Laws, form a complete description of classical physics.

12. Electromagnetic Waves

The power of Maxwell's equations was that they predicted that a travelling wave composed of oscillations in electric and magnetic fields, more than a decade before such waves were generated by Heinrich Hertz. The possibility that light might be such a wave is strengthened by the numerical "coincidence":

$$\frac{1}{\mu_0\,\varepsilon_0} = \frac{4\pi \text{ x} 9 \text{x} 10^9}{4\pi \text{ x} 10^{-7}} = 9 \text{x} 10^{16} = c^2$$

where c is the velocity of light, 3 x 10^8 ms^{-1}.

Equations II and III show an interdependence between B and E. We have:

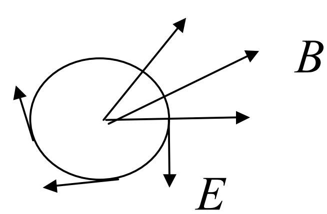
$$\oint \mathbf{E}.d\mathbf{l} = -\frac{d}{dt}\oint_S \mathbf{B}.d\mathbf{S}$$

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0\varepsilon_0\frac{d}{dt}\oint_S \mathbf{E}.d\mathbf{S}$$

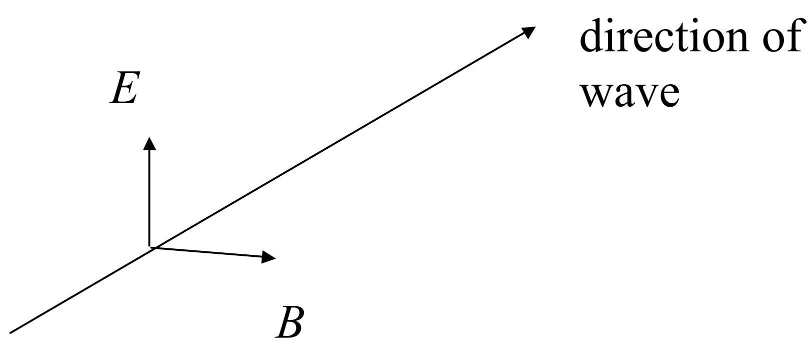
The symmetry in these equations is apparent. A change in B causes an E, a change in which in turn causes a B, and so on.

The full proof the Maxwell's equations permit a wave-like solution requires the conversion of the equations into vector-differential form and the mathematics is beyond the scope of this course.   An outline of the procedure (making several gratuitous assumptions) is given below.

First of all we note that the above two equations imply that the electric and magnetic fields are perpendicular to each other.

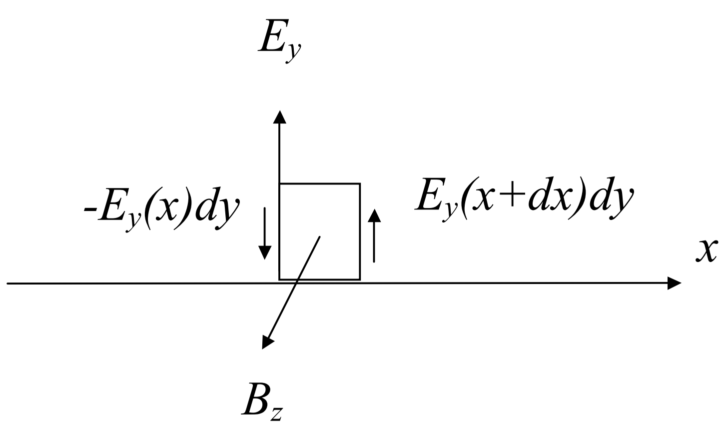


Since the equations are symmetric in *B* and *E*, we can assume that neither are privileged and so that the direction of propagation of the wave is perpendicular to both. Electromagnetic waves are therefore transverse waves.



Next, we need to get rid of the surface and line integrals.

Consider a small square loop of side *dx, dy* placed at a point on the wave.



Recalling that as $dx \to 0$, from the definition of a derivative:

$$\frac{dE_y}{dx} = \frac{E_y\left(x + dx\right) - E_y\left(x\right)}{dx}$$

the line integral of E_y round a loop in the *xy*-plane will be:

$$\oint \mathbf{E}.d\mathbf{l} = \frac{dE_y}{dx}\,dx\,dy$$

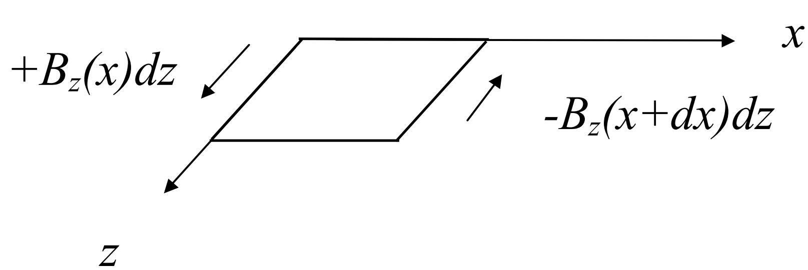
On the other hand, B_z only has a component in the xy-plane. While this might change along dx, its average value will be B_z. Hence:

$$\oint_S \mathbf{B}.d\mathbf{S} = B_z dxdy$$

Thus

$$\frac{dE_y}{dx} = -\frac{dB_z}{dt}$$

A similar argument can of course be made for the line integral of **B** and the surface integral of **E**, except that a negative sign enters in the former because of the geometry:



Hence:

$$\frac{dB_z}{dx} = -\mu_0\varepsilon_0 \frac{dE_y}{dt}$$

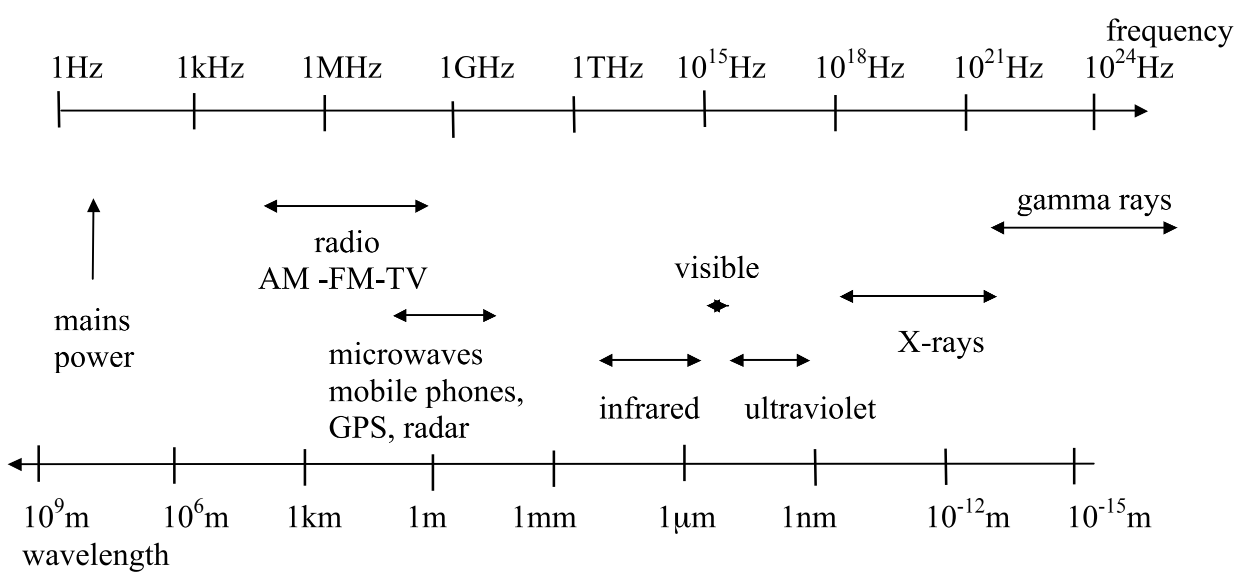
These two equations can be combined to give:

$$\frac{d^2E_y}{dx^2} = -\mu_0\varepsilon_0 \frac{d^2E_y}{dt^2}$$

$$\frac{d^2B_z}{dx^2} = -\mu_0\varepsilon_0 \frac{d^2B_z}{dt^2}$$

showing that both the electric and magnetic fields can have a wave motion with velocity $1/\sqrt{\mu_0\varepsilon_0}$

Today we know that a large variety of apparently distinct phenomena are in fact electromagnetic waves of different frequency:



13. Electromagnetism

The wave equation for electromagnetic waves provided the basis for Einstein's special theory of relativity, since it predicts that all observers (who measure the same value of ε_0) will measure the same value for the speed of light, irrespective of their velocity.

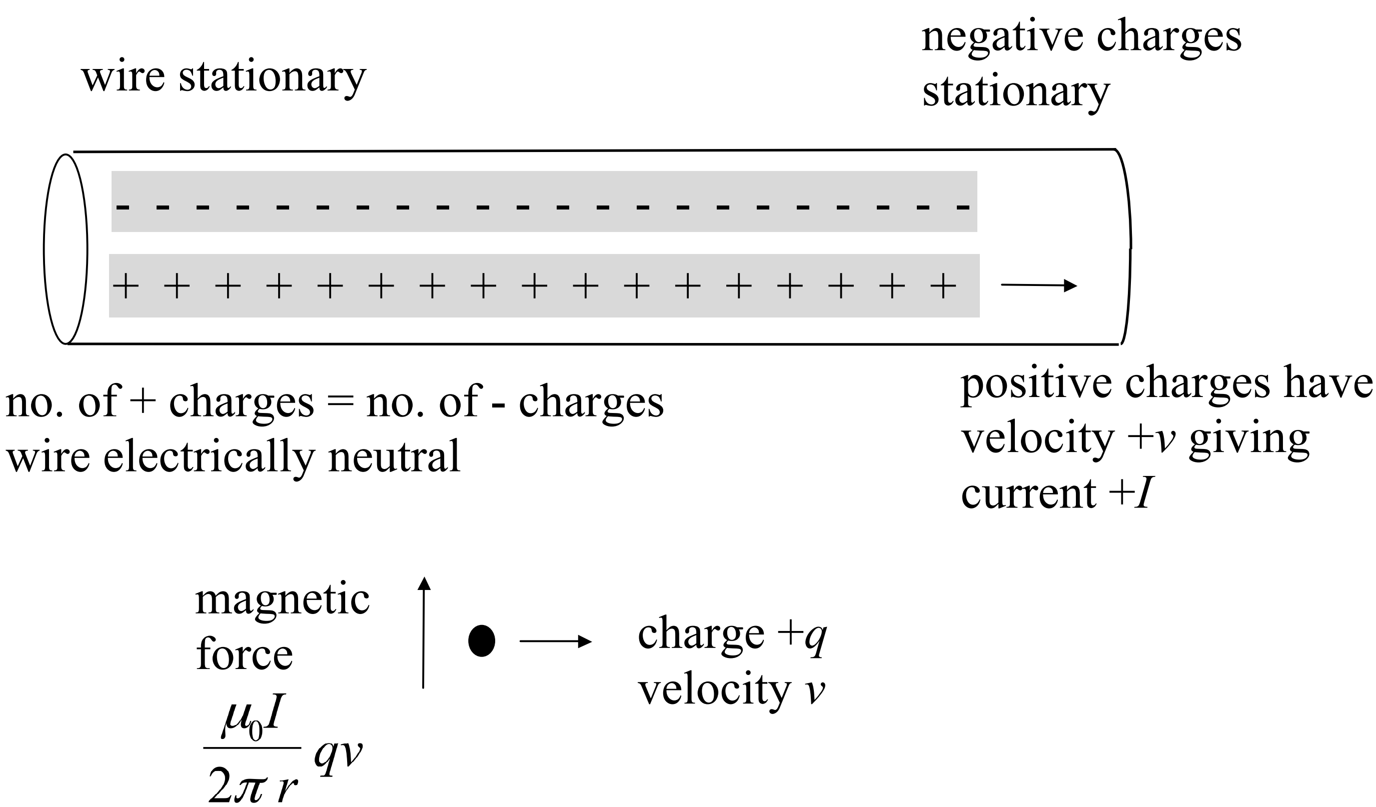
(In any case: velocity with respect to what?)

However there is another interesting consequence of special relativity, namely that the magnetic force does not exist as a separate entity but is just a relativistic "by-product" of the electric force.

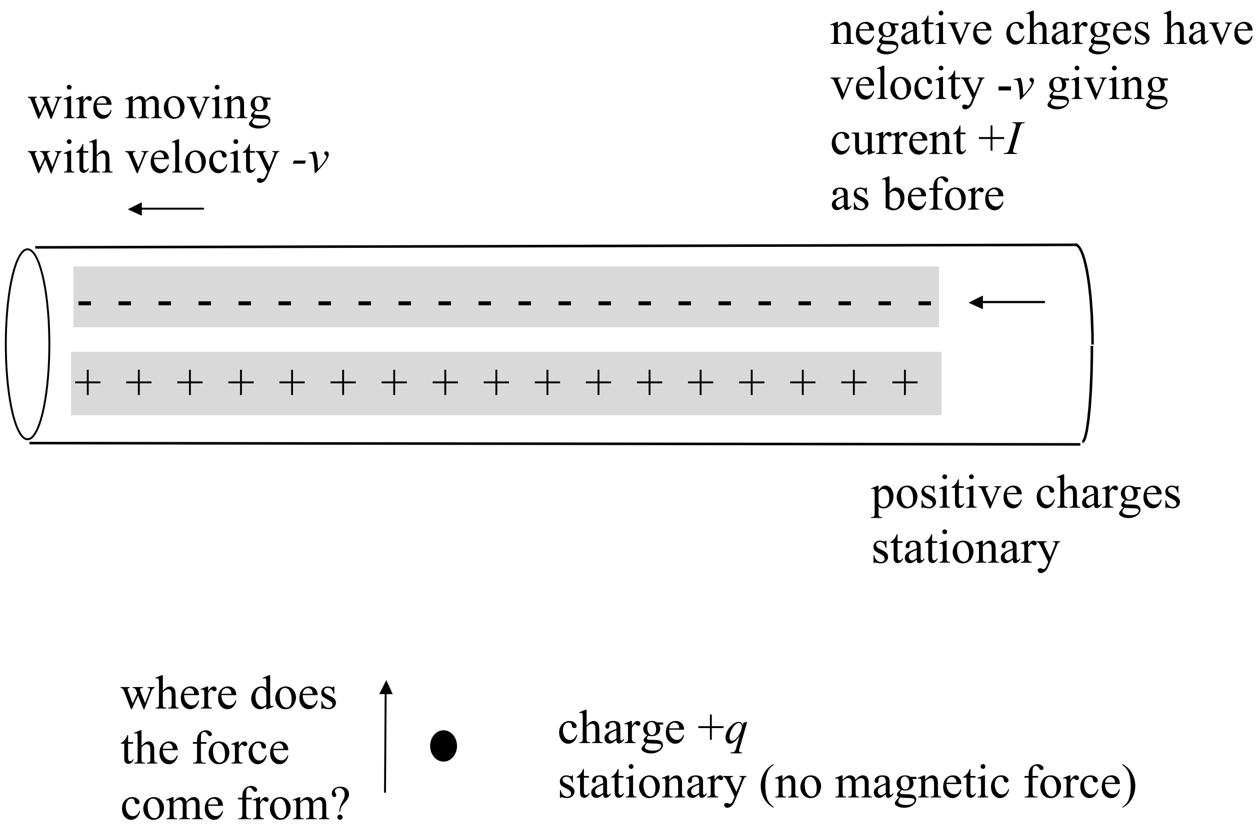
Consider a neutrally-charged wire carrying a current because of moving positive charges within it.   A moving positive charge, some distance from the wire, will feel no net electric force, but a magnetic force.

For simplicity, suppose that the velocity of the charge is equal to the velocity of the positive charges inside the wire.

From the point of view of a stationary observer:

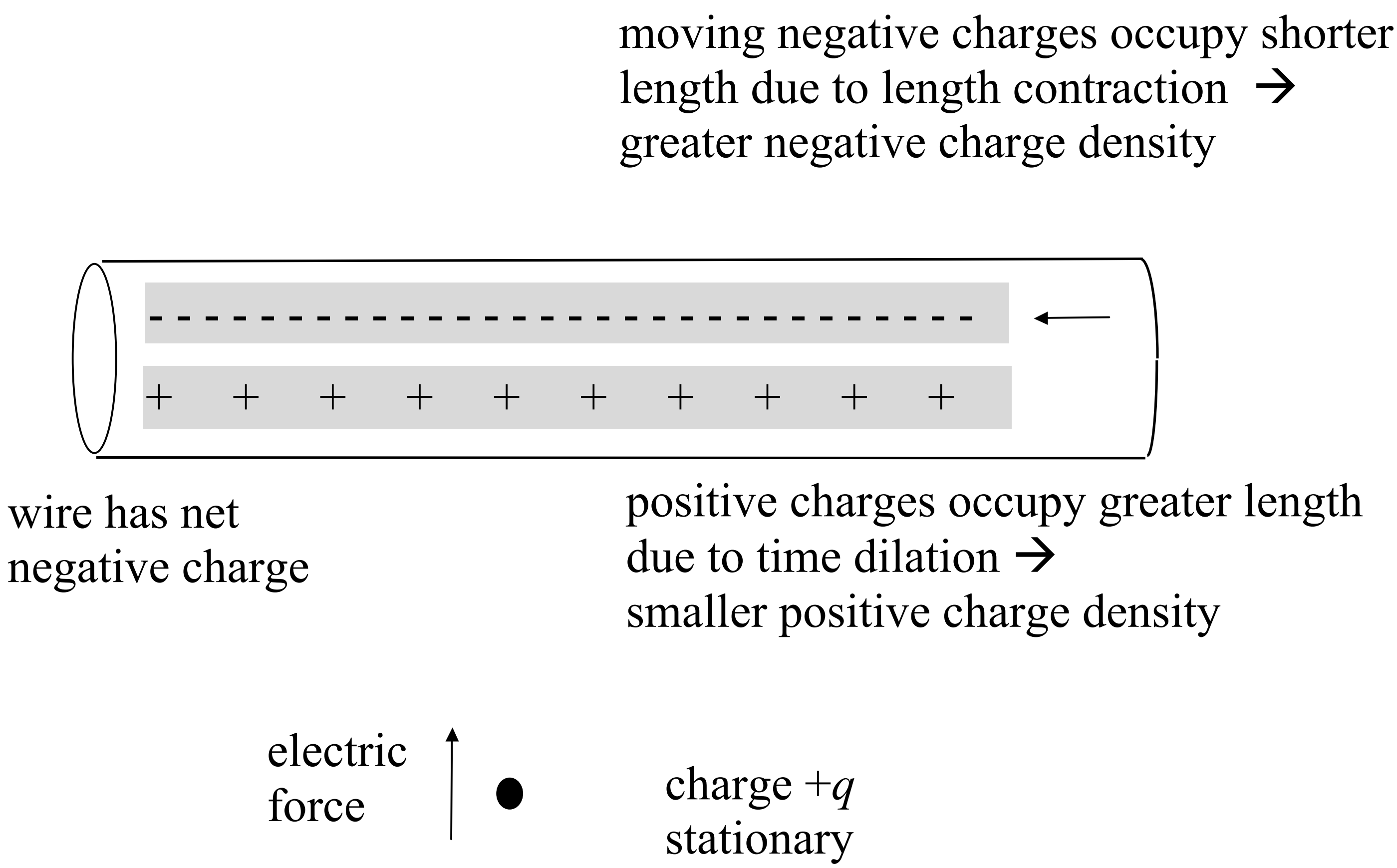


The situation is very different from the point of view of an observer travelling with the velocity of the charge. Since the charge is stationary in this frame, there cannot be any magnetic force!



Yet there *must* be a force on the charge since the situation must look similar to both observers.

Relativity comes to the rescue! Length contraction and time dilation means that the wire now appears charged. The charge has an *electric* force on it that can be shown to be exactly equal to the magnetic force seen in the stationary frame.



This argument completes the unification of the electric and magnetic forces into different aspects of just one force.